BIOS 6244 Analysis of Categorical Data
September 26, 2005 Lecture
Assignment 1 (due October 5, 2005)

Confidence Interval for a Proportion (pp. 11-12)

It is usually not enough to simply conclude that $\pi \neq \pi_0$ or that $\pi = \pi_0$ cannot be rejected. We almost always want a confidence interval CI(π) as well.

<u>Approximate Method 1</u>

In general, an approximate 100(1-α)% CI(θ), where θ is any unknown parameter, is given by

$$\hat{\theta} \pm z_{\alpha/2} SE(\hat{\theta})$$

where $z_{\alpha/2}$ denotes the upper α/2-percentage point of the standard normal distribution.

In health sciences research, a 95% C.I. is always used, so $z_{\alpha/2} = z_{.025} = 1.96$.

<u>Example</u>

For the binomial,

$$95\%\ \text{CI}(\pi) = \text{p} \pm 1.96\sqrt{\frac{\text{p(1-p)}}{\text{N}}} \qquad (1)$$

Note that a different formula is being used for $SE(\hat{\theta})$ from what we used when performing an approximate hypothesis test for π. The formula we used before was

$$SE(\hat{\theta}) = \sqrt{\frac{\pi_0(1-\pi_0)}{N}}. \qquad (2)$$

(See notes from August 24, 2005). Equation (2) above is the formula one should use for $SE(\hat{\theta})$ when assuming that the null hypothesis is true, i.e., that $\pi = \pi_0$.

When finding a confidence interval, we need a formula for $SE(\hat{\theta})$ that we can use regardless of whether the null hypothesis is true, so we use the maximum likelihood (ML) estimate p in place of π_0:

$$SE(\hat{\theta}) = \sqrt{\frac{p(1-p)}{N}}.$$

For our sample, N = 10, $p = \frac{3}{10} = .3$. So, substituting into Equation (1), we obtain the following 95% CI(π):

$$p \pm 1.96\sqrt{\frac{p(1\text{-}p)}{N}} = .3 \pm 1.96\sqrt{\frac{.3(1\text{-}.3)}{10}} = .3 \pm .28 = (.02, .58)$$

Note that the conclusion from this confidence interval is NOT consistent with the conclusion from the hypothesis test of H_0: $\pi = .1$ that we conducted during the lecture on August 24. Since .1 is contained in the interval (.02, .58), we fail to reject H_0. But our approximate p-value from our previous hypothesis test was .035, indicating that we should reject H_0.

Approximate Method 2

A better way to construct confidence intervals for parameters of discrete distributions is to define

$$95\%\ CI(\theta) = \{\text{all values } \theta_0 \mid H_0: \theta = \theta_0 \text{ is not rejected using the sample data}\}.$$

In Exercise 1.15, p. 15 of our textbook, Agresti describes a method in which, given π_0, N, and y, one can find π_L and π_U such that all values π_0 between π_L and π_U would NOT be rejected when testing H_0: $\pi = \pi_0$ using the approximate test with $\alpha = .05$. Therefore, (π_L, π_U) is an approximate 95% CI(π). The general approach described by Agresti in Exercise 1.25 is called *Fieller's method*.

See the Example worked out by Agresti at the top of p. 12 in our textbook.

For our sample data, N = 10, y = 3, $\pi_0 = .1$. Then an approximate 95% CI(π) using Fieller's method is (.11, .60). Note that this is consistent with the hypothesis test we conducted on August 24 since .1 is not contained in this interval.

Exact Method

The approach here is the same as with Fieller's method except that we seek π_L and π_U such that for all $\pi_L < \pi_0 < \pi_U$, H_0: $\pi = \pi_0$ would not be rejected using the EXACT hypothesis test with $\alpha = .05$. There are various methods that could be used to find π_L and π_U, all of which are beyond the scope of this course and all of which give about the same answer. One of the better known methods is due to Pearson and Clopper and is implemented in SimCalc and many other statistical packages. For our sample, this method yields an EXACT 95% CI(π) of (.07, .65). Note that this interval includes .1, so the statistical inference from the exact C.I. is consistent with that of the exact test of H_0: $\pi = \pi_0$, which yielded p = .140. (See lecture notes from August 24, 2005.)

A summary of the results from each of the methods we have examined is given below:

Method	**p-value for test of H_0: $\pi = .1$**	**95% C.I. (π)**
normal approximation	.035*	---
“usual” C.I.	---	(.02, .58)
“Fieller’s method” C.I.	---	(.11, .60)*
Exact	.140	(.07, .65)

*Statistically significant result

Discussion

It is obvious from today’s lecture that the “usual” methods of finding a CI(π) and for testing an hypothesis concerning π can give misleading results, especially if N is small and p (or π_0) is near 0 or 1. Given modern computing capabilities, there is no valid reason not to use exact methods, regardless of the value of N. Some books recommend that exact methods should be used only when N is small, and this advice was appropriate when computing exact binomial (or Poisson) probabilities was extremely time-consuming. This is no longer an issue of concern for modern-day statisticians.

It is also important to choose confidence interval and hypothesis testing methods that yield consistent results. It causes no small amount of consternation and confusion to a clinical investigator when the results indicate that $p < .05$, but the confidence interval includes the null value. The idea that confidence intervals and hypothesis tests are ALWAYS supposed to yield the same conclusion may be one of the few things that the investigator remembers from their basic biostatistics course (which may have consisted of only 2 or 3 lectures), so they have difficulty understanding how this could not be the case for their set of data. Whenever possible, statisticians should always use statistical approaches that will yield the same conclusion for an hypothesis test and the corresponding confidence interval method when applied to the same set of data.